Explicit Methods for Modularity of K3 Surfaces and Other Higher Weight Motives
Institute for Computational and Experimental Research in Mathematics
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Motivation: equidistribution for $L$-functions

For a motive $M$ (with $\mathbb{Q}$-coefficients), consider its $L$-function in the analytic normalization:

$$L(s) = \prod_p L_p(s) = \prod_p F_p(p^{-s})^{-1}, \quad F_p(T) = 1 - a_p T + \cdots.$$ 

Conjecture (generalized Sato-Tate conjecture; Serre, 1994)

The polynomials $F_p(T)$ are equidistributed for the image of Haar measure (via the characteristic polynomial map) on a specified compact Lie group $\text{ST}(M)$ (the Sato-Tate group).

E.g., the $a_p$ vary like traces of random matrices in $\text{ST}(M)$.

Proposition

For any given degree, weight, and Hodge numbers (i.e., Gamma factors), there are only finitely many possible Sato-Tate groups.
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For any given degree, weight, and Hodge numbers (i.e., Gamma factors), there are only finitely many possible Sato-Tate groups.
Example: elliptic curves

Take $M = H^1(E)$ with $E$ an elliptic curve over $\mathbb{Q}$.

If $E$ has CM, then $\text{ST}(M)$ is the normalizer of $\text{SO}(2, \mathbb{R})$ in $\text{SU}(2)$:

http://math.mit.edu/~drew/g1_D2_a1f.gif.

Equidistribution follows easily from CM theory (Hecke).

If $E$ has no CM, then $\text{ST}(M) = \text{SU}(2)$:

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Equidistribution (i.e., the original Sato-Tate conjecture) is known but hard: it uses potential modularity of symmetric power $L$-functions (Taylor et al.).

If we consider $E$ over a number field $K$, then the CM picture changes if the CM field is contained in $K$, as $\text{ST}(M)$ decreases to $\text{SO}(2, \mathbb{R})$:

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Overview

More examples to consider

For the rest of the talk, we will be interested in the following three classes of motives. Here $K$ denotes an arbitrary number field (but you may assume $K = \mathbb{Q}$), $w$ is the motivic weight, $(h^0,w,\ldots,h^w,0)$ is the Hodge vector, and $d = \sum_{p+q=w} h^p,q$ is the degree of the associated $L$-function.

- $M$ has weight 1 and Hodge vector $(g,g)$. This means that $M = H^1(A)$ for $A/K$ an abelian variety of dimension $g$.
- $M$ has weight 2 and Hodge vector $(1,20,1)$. In particular, we want $M = H^2(X)$ for $X/K$ a K3 surface.
- $M$ has weight 3 and Hodge vector $(1,1,1,1)$, e.g., a hypergeometric motive from the Dwork pencil

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = \lambda x_0 x_1 x_2 x_3 x_4.$$

\[\text{To force this, we must fix some extra data, e.g., the intersection pairing and the ample cone.}\]
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The Betti-Hodge realization and the Mumford-Tate group

Fix an embedding $K \hookrightarrow \mathbb{C}$. Let $V$ denote the Betti (singular) cohomology of $M$ with $\mathbb{Q}$-coefficients; then $\dim_{\mathbb{Q}} V = d$.

The duality $M \times M \to \mathbb{Q}(-w)$ induces a perfect bilinear pairing $\psi$ on $V$. Let $\text{Gl}_\text{iso}(V, \psi)$ be the associated group of symplectic (if $w$ is odd) or orthogonal (if $w$ is even) similitudes.

The space $V_{\mathbb{C}} = V \otimes_{\mathbb{Q}} \mathbb{C}$ admits a canonical Hodge decomposition $\bigoplus_{p+q=w} V^{p,q}$ with $\dim_{\mathbb{C}} V^{p,q} = h^{p,q}$. Let

$$\mu_{\infty, V} : \mathbb{G}_m(\mathbb{C}) \to \text{GL}(V_{\mathbb{C}})$$

be the cocharacter acting with weight $-p$ on $V^{p,q}$.

The Mumford-Tate group of $M$ is the minimal (connected) $\mathbb{Q}$-algebraic subgroup $\text{MT}(M)$ of $\text{Gl}_\text{iso}(V, \psi)$ through which $\mu_{\infty, V}$ factors.
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For $n$ a positive integer for which $wn$ is even, put $p = wn/2$ and

$$(V \otimes^n)^{p,p} := (V_{\mathbb{C}} \otimes^n)^{p,p} \cap V \otimes^n.$$ 

Then $\text{MT}(M)$ can also be characterized as the maximal subgroup of $\text{Glso}(V, \psi)$ fixing $(V \otimes^n)^{p,p}$ for all $n$. 
Another characterization of the Mumford-Tate group

The *Mumford-Tate group* of $M$ is the minimal (connected) $\mathbb{Q}$-algebraic subgroup $\text{MT}(M)$ of $\text{Glso}(V, \psi)$ through which $\mu_\infty, V$ factors. For $n$ a positive integer for which $wn$ is even, put $p = wn/2$ and

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Then $\text{MT}(M)$ can also be characterized as the maximal subgroup of $\text{Glso}(V, \psi)$ fixing $(V \otimes^n)^{p,p}$ for all $n$. 
Under the Hodge conjecture\(^2\), \((V \otimes n)^{p,p}\) is spanned by the Chern classes of algebraic cycles defined over \(\overline{K}\). We thus have an action of the absolute Galois group \(G_K\) on \((V \otimes n)^{p,p}\).

The **motivic Galois group** \(\text{Gal}(M)\) is the subgroup of \(g \in \text{Glso}(V, \psi)\) for which there exists \(\tau = \tau(g) \in G_K\) such that the actions of \(g\) and \(\tau\) on \((V \otimes n)^{p,p}\) coincide for all \(n\). By construction, we have an exact sequence

\[ 1 \to \text{Gal}(M)^{\circ} = \text{MT}(M) \to \text{Gal}(M) \to \text{Gal}_{L/K} \to 1 \]

of algebraic groups over \(\mathbb{Q}\), where \(L\) is some finite extension of \(K\). (Here and throughout, \(G^{\circ}\) denotes the maximal connected subgroup of \(G\).)

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The motivic Galois group

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The Sato-Tate group

Define the *algebraic Sato-Tate group*

\[ \text{AST}(M) = \text{Gal}(M) \cap \text{Glso}(V, \psi)^{\circ}; \]

note that \( \text{Glso}(V, \psi)^{\circ} \) equals \( \text{Sp}(V, \psi) \) or \( \text{SO}(V, \psi) \).

Again by construction, we have an exact sequence

\[ 1 \rightarrow \text{AST}(M)^{\circ} \rightarrow \text{AST}(M) \rightarrow \text{Gal}_{L/K} \rightarrow 1 \]

of algebraic groups over \( \mathbb{Q} \) (for the same \( L \)).

The *Sato-Tate group* \( \text{ST}(A) \) is a maximal compact subgroup of \( \text{AST}(M)_{\mathbb{C}} \).

We have an exact sequence of compact Lie groups

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Example in weight 1: abelian varieties [FKRS]

Endomorphisms and Sato-Tate groups

Put $M = H^1(A)$ for $A/K$ an abelian variety of dimension $g > 0$. Then

$$\text{Gl}_0(V, \psi) \cong \text{GSp}(2g) \quad \text{and} \quad (V \otimes 2)^{1,1} \cong \text{End}(A_K)_\mathbb{Q}.$$  

In many cases (e.g., when $g \leq 3$), the map

$$((V \otimes 2)^{1,1})^n \to (V \otimes 2^n)^{n,n}$$

is surjective, so $\text{AST}(M)$ and $\text{ST}(M)$ are determined entirely by endomorphisms. In these cases, the exact sequence

$$1 \to \text{ST}(M)^\circ \to \text{ST}(M) \to \text{Gal}_{L/K} \to 1$$

implies that $L$ is the minimal field for which $\text{End}(A_L) = \text{End}(A_K)$ (otherwise $L$ may be larger).
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Warmup: elliptic curves

If $A = E$ is of dimension $g = 1$, then

$$\text{Glso}(V, \psi) \cong \text{GL}(2) \quad \text{and} \quad (V \otimes^2)^{1,1} \cong \text{End}(E_K)_\mathbb{Q}.$$ 

- If $E$ has no CM, then $\text{AST}(M) = \text{SL}(2)$ and $\text{ST}(M) = \text{SU}(2)$.
- If $E$ has CM in $K$, then $\text{AST}(M)$ is the norm torus for $F/\mathbb{Q}$, where $F$ is the field of complex multiplication, and $\text{ST}(M) = \text{SO}(2, \mathbb{R})$.
- If $E$ has CM in an overfield $L/K$, then $\text{ST}(M)$ is the normalizer of $\text{ST}(M_L) = \text{SO}(2, \mathbb{R})$ in $\text{SU}(2)$.

This illustrates a general phenomenon: for fixed parameters, there are generally infinitely many options for the $\mathbb{Q}$-algebraic group $\text{AST}(M)$. By contrast, $\text{ST}(M)$ depends only on $\text{AST}(M)_\mathbb{R}$, for which there are only finitely many options.
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$$\text{Glso}(V, \psi) \cong \text{GL}(2) \quad \text{and} \quad (V \otimes^2)^{1,1} \cong \text{End}(E_K)_{\mathbb{Q}}.$$ 

- If $E$ has no CM, then $\text{AST}(M) = \text{SL}(2)$ and $\text{ST}(M) = \text{SU}(2)$.
- If $E$ has CM in $K$, then $\text{AST}(M)$ is the norm torus for $F/\mathbb{Q}$, where $F$ is the field of complex multiplication, and $\text{ST}(M) = \text{SO}(2, \mathbb{R})$.
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This illustrates a general phenomenon: for fixed parameters, there are generally infinitely many options for the $\mathbb{Q}$-algebraic group $\text{AST}(M)$. By contrast, $\text{ST}(M)$ depends only on $\text{AST}(M)_{\mathbb{R}}$, for which there are only finitely many options.
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Properties of Sato-Tate groups

For $M$ as above, the group $\text{ST}(M)$ satisfies the following conditions.

1. **(ST1)** The group $\text{ST}(M)$ is a closed subgroup of $\text{USp}(2g)$. (Equality is the generic case.)

2. **(ST2)** The connected group $\text{ST}(M)^\circ$ is the closure of the subgroup generated by **Hodge circles**: images of cocharacters $\theta : \text{U}(1) \rightarrow \text{ST}(M)^\circ$ with weight $p - q$ of multiplicity $h^{p,q}$.

3. **(ST3)** For each connected component $C$ of $\text{ST}(M)$ and each irreducible character $\chi$ of $\text{GL}(2g, \mathbb{C})$, the average of $\chi$ on $C$ is an integer.

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Mumford-Tate groups of abelian surfaces

**Theorem (well-known)**

For \( g = 2 \), there are exactly 6 conjugacy classes of subgroups of USp(4) which can occur as \( \text{ST}(A)^\circ \), isomorphic to

\[
\text{U}(1), \text{SU}(2), \text{U}(1) \times \text{U}(1), \text{U}(1) \times \text{U}(2), \text{U}(2) \times \text{U}(2), \text{USp}(4).
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This list corresponds to the possibilities for \( \text{End}(A_{\overline{K}})^{\mathbb{R}} \):

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Consequently, the passage from \( A \) to \( \text{ST}(A)^\circ \) conflates distinct geometric behaviors. For instance, a simple CM abelian fourfold gives the same group \( \text{U}(1) \times \text{U}(1) \) as the product of two nonisogenous CM elliptic curves, as in both cases \( \text{End}(A_{\overline{K}})^{\mathbb{R}} \cong \mathbb{C} \times \mathbb{C} \).
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Theorem ([FKRS])

Take $g = 2$.

(a) There are 55 conjugacy classes of subgroups of $\text{USp}(2g)$ satisfying $(ST1)$, $(ST2)$, $(ST3)$.

(b) Of these, exactly 52 are realized as $\text{ST}(M)$ for suitable $A$. The generic case $\text{ST}(M) = \text{USp}(4)$ occurs iff $\text{End}(A_K) = \mathbb{Z}$.

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For illustrated examples, see

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Consequences for abelian surfaces

For $g = 2$, we read off some arithmetic consequences.

**Corollary (improvement of a result of Silverberg)**

The minimal field $L/K$ with $\text{End}(A_L) = \text{End}(A_K)$ has degree dividing 48. This bound is realized even for $K = \mathbb{Q}$, e.g., by the Jacobian of $y^2 = x^6 - 5x^4 + 10x^3 - 5x^2 + 2x - 1$.

**Corollary**

The density of prime ideals with zero Frobenius trace exists and belongs to

$$\left\{0, \frac{1}{6}, \frac{1}{4}, \frac{1}{8}, \frac{1}{24}, \frac{1}{2}, \frac{1}{12}, \frac{1}{8}, \frac{1}{4}, \frac{1}{24}, \frac{1}{16}, \frac{1}{8}\right\}.$$

All of these cases are realized, e.g., $7/8$ by $y^2 = x^5 + 2x$. (Only the case $3/8$ cannot occur for $K = \mathbb{Q}$.)
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Higher-dimensional abelian varieties

For $g \geq 3$, it seems difficult to get a complete classification. Most of the cases occur when $\text{ST}(M)^{\circ}$ is a one-dimensional torus; these cases occur for twisted powers of CM elliptic curves.

By contrast, suppose that $M$ is discrete in the sense of Gross’s lecture, i.e., the centralizer of $\text{ST}(M)^{\circ}$ in $\text{USp}(2g)$ is finite. Then one gets a finite list of options even without $(ST3)$. One only needs to describe the subgroups of the group $\text{Out} (\text{ST}(M)^{\circ})$; that group consists (approximately) of automorphisms of the Dynkin diagram.
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4 Example in weight 2: K3 surfaces [?

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6 References
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Take $M = H^2(X)$ for $X/K$ a K3 surface.

Recall that to compute $\text{ST}(M)$, we have to look at $(V \otimes n)^{p,p}$ whenever $n > 0$, $nw$ is even, and $p = nw/2$. For $n = 1$, this is $\text{NS}(X_K)_\mathbb{Q}$ by the Lefschetz $(1,1)$ theorem.

Put

$$\rho = \text{rank } \text{NS}(X), \quad \bar{\rho} = \text{rank } \text{NS}(X_K).$$

Then

$$\text{ST}(M) \subseteq \text{SO}(22 - \rho), \quad \text{ST}(M)^\circ \subseteq \text{SO}(22 - \bar{\rho})$$

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As usual, $\text{ST}(M)^\circ$ is determined by $\text{MT}(M)$. Luckily, K3 surfaces do not exhibit the subtleties associated to Mumford-Tate groups of abelian varieties: $\text{ST}(M)^\circ$ is “as large as possible” (ultimately because $h^{2,0} = 1$).

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Let $V_{tr}$ be the orthogonal complement of $V^{1,1}$ in $V$.

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Aside: the Mumford-Tate conjecture holds for $X$ (Tankeev, 1995).
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As usual, $\text{ST}(M)^\circ$ is determined by $\text{MT}(M)$. Luckily, K3 surfaces do not exhibit the subtleties associated to Mumford-Tate groups of abelian varieties: $\text{ST}(M)^\circ$ is “as large as possible” (ultimately because $h^{2,0} = 1$).

**Theorem (Zarhin, 1983; [Z])**

Let $V_{tr}$ be the orthogonal complement of $V^{1,1}$ in $V$.

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For $X$ the Kummer of an abelian surface $A$, we have

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How does this relate to Zarhin’s theorem?

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<thead>
<tr>
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<th>$\bar{\rho}$</th>
<th>$E_R$</th>
</tr>
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<tbody>
<tr>
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<td>20</td>
<td>$C$</td>
</tr>
<tr>
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<td>$SO(3)$</td>
<td>19</td>
<td>$R$</td>
</tr>
<tr>
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<td>18</td>
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Recover the classification of Sato-Tate groups of abelian surfaces. Do non-Kummer surfaces with $\bar{\rho} = 18$ account for the 3 missing groups?
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Using Zarhin’s theorem, classify the possible Sato-Tate groups associated to K3 surfaces of arbitrary rank. In particular, what are the possible zero trace densities besides 0, 1/2?

Note that given $\bar{\rho}$ and $E_\mathbb{R}$, $\text{ST}(M)$ is determined by its action on $\text{NS}(X_K)_\mathbb{R}$ (because $\text{ST}(M)^\circ$ is “as large as possible”).

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2 Construction of the Sato-Tate group [S, BK1, BK2]

3 Example in weight 1: abelian varieties [FKRS]

4 Example in weight 2: K3 surfaces [?]

5 Example in weight 3: hypergeometric motives [FKS]

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We will need the following constructions:

- A direct sum of a weight 2 newform and a weight 4 newform.
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Theorem ([FKS])

Take $M$ as above.

(a) There are 26 conjugacy classes of subgroups of $\text{USp}(4)$ satisfying $(ST1), (ST2), (ST3)$.

(b) Of these, at least 25 are realized as $\text{ST}(M)$ for suitable $M$.

Due to the changed position of the Hodge circles, the options for $\text{ST}(M)^{\circ}$ are not the same as for abelian surfaces:

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Taxonomy of sources

Let us explain how these groups arise from our examples. In all cases, the upper bound is achieved by a “generic” example.

- A direct sum of a weight 2 newform and a weight 4 newform: $\text{ST}(M)^{\circ} \subseteq \text{SU}(2) \times \text{SU}(2)$. We also see $\text{U}(1) \times \text{U}(1)$ and $\text{U}(1) \times \text{SU}(2)$.

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Degenerations of Sato-Tate groups

Over $\overline{\mathbb{Q}}$, the $j$-line contains infinitely many CM points; similarly, any positive-dimensional family of weight 1 motives contains infinitely many special subvarieties of codimension 1 where the Sato-Tate group drops (as in the André-Oort conjecture).

By contrast, for motives of weight greater than 1, a Hodge structure cannot vary arbitrarily in families; its variation is constrained by Griffiths transversality (thus precluding a universal family). Refining a prediction of de Jong, the generalized André-Oort conjecture (see Klingler’s AMS SLC 2015 lecture) suggests that jumping can only occur on a Zariski dense subset if the family “arises from a Shimura variety.”

This is consistent with our experimental data: in the Dwork pencil, one expects that over all $K$, only finitely many fibers have $\text{ST}(M) \neq \text{USp}(4)$. Over $\mathbb{Q}$, we found no such examples (excluding the Fermat fiber).
Degenerations of Sato-Tate groups

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By contrast, for motives of weight greater than 1, a Hodge structure cannot vary arbitrarily in families; its variation is constrained by Griffiths transversality (thus precluding a universal family). Refining a prediction of de Jong, the generalized André-Oort conjecture (see Klingler’s AMS SLC 2015 lecture) suggests that jumping can only occur on a Zariski dense subset if the family “arises from a Shimura variety.”

This is consistent with our experimental data: in the Dwork pencil, one expects that over all \( K \), only finitely many fibers have \( \text{ST}(M) \neq \text{USp}(4) \). Over \( \mathbb{Q} \), we found no such examples (excluding the Fermat fiber).
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